The Scaling Theory: VII. The Scaling Transformations of Second Type

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13. The Scaling Transformations of the Second Type

In deriving the scaling transformations of the first type, the geometric duration and length, T and R, of the virtual light trip ($B\ when\ b\ \to O\ and\ o$) in the timed inertial S, were adopted as the "*standard reference characters*". From these reference characters were induced corresponding geometric duration and length, t and r in s, such that

(i)-The velocity of light is also c in s.

(ii)-Both sets of geometric time and distance intervals are real (actual) in S as well as in s. From the latter requirement we deduced that every relation imposed in S on t and r generates a dual relation imposed in s on T and R, and consequently, on t and r.

The second type of scaling transformations differ from the first type in that, the standard reference characters are different.

Derivation

Assume that at instant $T = 0$ in S, which corresponds to ($o \in s\ is\ at\ O \in S$) and ($b \in s\ is\ at\ B \in S$), our source of light b which is stationary in s emits a pulse of light towards the point $o \in s$. In the timed frame S, the emitted pulse targets the point $o \in s$ which was contiguous to O at the instant of emission, but occupies when light is received a new position, $o' \in S$. The characters of the *hypothetical reference trip* ($B \to O$) in S, though never occurred, are known from start, and accordingly are imposed as the reference characters from which the characters of the true trip derive.

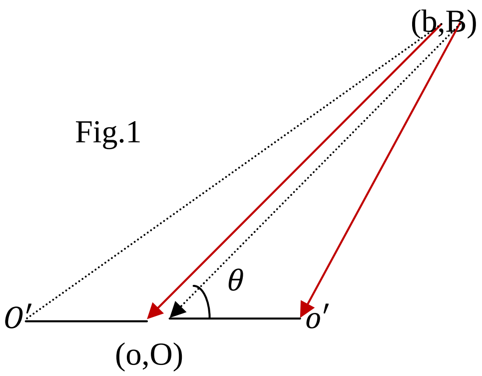

Fig.1

Fig.1 shows the light's paths as observed in S and s.

At an initial time $T = 0$, the S observers associate with the geometric distance R between the location of the source and its final destination o which is at O (at $T = 0$) the geometric duration T, such that

(13.1) $$R = cT.$$

The last equation relates the geometric duration of the hypothetical trip ($B \to O$), that never took place, to its length. The characters R and T of the hypothetical trip ($B \to O$) are adopted as the *standard reference characters*. We endow now the frame s with geometric distance and time intervals that render the velocity of light within s is c. Since the source of light b is stationary in s, we impose on the geometric length r and duration t of the trip ($b \to o$) to satisfy the equation

(13.2) $$r = ct.$$

The geometric characters of this trip must be as real in *S* as they are in *s*. In the *S* frame thus, the pulse emanating from (*b* at *B*) takes to reach *o*, which is no more

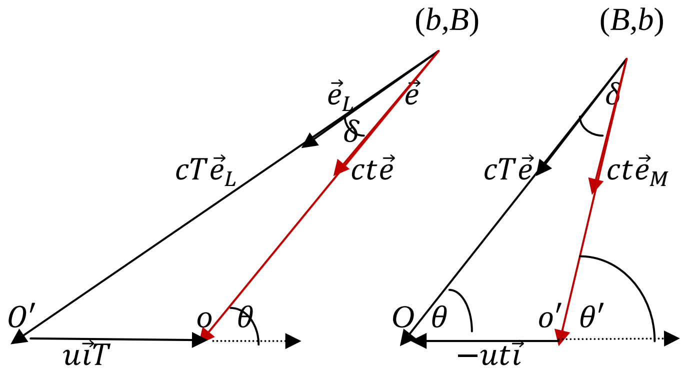

Fig.2.The view of *s* Fig.3 The view of *S*

at *O*, a period t during which *o* is displaced to a point o', where $\overrightarrow{Oo'} = ut\vec{\imath}$. Applying tentatively the displacement law

$$\overrightarrow{BO} = \overrightarrow{Bo'} + \overrightarrow{o'O} = \overrightarrow{bo'} + \overrightarrow{o'O},$$

we obtain

(13.3) $$cT\vec{e} = (c\vec{e}_M - u\vec{\imath})t,$$

where $\overrightarrow{BO} = R\vec{e}$, and $\overrightarrow{bo'} = r\vec{e}_M$ since the light trip $(b \to o')$ is realized in *s* as the trip $(b \to o)$. In order to put the induced time *t* in *s* on equal footing with the time *T* in the timed frame *S*, the following condition must hold: were *s* the timed inertial frame and $(B \to O)$ the true trip, then the induced time in *S* should be its intimate geometric time *T* . Now the light emitted from (*B at b*) towards (*O when at o*) is seen in *s* ejected in the direction $\vec{e}_L$ which is the unit of the vector $\overrightarrow{bO'} = R\vec{e}_L = cT\vec{e}_L$. Light reaches *O*, when *O* is at a point $O' \in s$, where $\overrightarrow{oO'} = -uT\vec{\imath}$. Using tentatively the law of displacements addition

$$\overrightarrow{bo} = \overrightarrow{bO'} + \overrightarrow{O'o} = \overrightarrow{BO'} + \overrightarrow{O'o},$$

yields

(13.4) $$ct\vec{e} = (c\vec{e}_L + u\vec{\imath})T,$$

since $(B \to O')$ is realized in *S* as the trip $(B \to O)$. From (13.3) and (13.4) we have

(13.5) $$\Gamma_2 \equiv \frac{t}{T} = \left|\frac{c\vec{e}_L + u\vec{\imath}}{c\vec{e}_M - u\vec{\imath}}\right| \frac{1}{\Gamma_2}.$$

Comparing with (4.5) yields

(13.6) $$\Gamma_2(u,\theta) = 1/\Gamma(u,\theta),$$

and the *scaling transformations of the second type* take the form

(13.7) $$t = \Gamma_2(u,\theta)T = T/\Gamma(u,\theta), \qquad r = \Gamma_2(u,\theta)R = R/\Gamma(u,\theta).$$

Note that θ in *s* is the angle between the ray and the velocity of *S* in *s*. It is also the angle in *S* between the initial radius vector of the source's position and the velocity of *o* (or the velocity of *s*) in *S*.

The latter transformations *determine the length r and the duration t of the true trip* $(b \to o')$, <u>*as it is observed in S*</u>, *in terms of the geometric length R and duration T of a hypothetical trip* $(B \to O)$, *which are already known in S. Since the trip* $(b \to o')$ *is realized in s as the trip* $(b \to o)$, <u>*r and t are also the length and duration of the latter trip in s.*</u> We shall call *r* and *t the proper length and duration* of the light trip $(b \to o')$. Equally, the latter transformations determine the length *R* and duration *T* of a true light's trip $(B \to O')$, as observed in a timed inertial frame *s*, in terms of the geometric length and duration of a hypothetical trip $(b \to o)$, which are known in s in this case. Since the trip $(B \to O')$ is realized in *S* as the trip $(B \to O)$, *R* and *T* are also the length and duration of the latter trip in *S.* Thus *R* and *T* in this case are the *proper characters* of the true trip $(B \to O')$.

In case *b* is the source, the transformations (13.7) are written in the form

(13.8) $$t = R/c\Gamma(u,\theta), \qquad r = R/\Gamma(u,\theta),$$

which yields the following important result: *the coordinates in the timed inertial frame S are sufficient to determine the length and duration of the given trip. Alternatively, the system of clocks in the timed inertial frame S alone is sufficient to determine the length and duration of any trip within s, as well as in any other inertial frame.* Parallel results hold in *s* were *B* the true source.

It is evident that the paths of the true and hypothetical trips are distinct in this case, and accordingly the directional angles are distinct too. From the geometry of the figures 2 and 3, and if the angle $\angle(\overrightarrow{OB}, OX)$ is measured θ in *S*, then the *s* observers measure the same value between the negative direction of the true trip and the *x*-axis. Using figure 3 we find that the *S* observers assign to the angle between the negative direction of the true trip and the *X*-axis the angle

(13.9) $$\theta' = \theta + \delta,$$

where

(13.10) $$sin\delta = \left(\frac{v}{c}\right) sin\theta.$$

The Standard Reference Characters

It is constructive at this stage to specify the *standard reference characters* in both types of transformations. When deriving the scaling transformations of the first type with b is a true source, the geometric characters of the virtual trip $(B\ when\ b \to O\ and\ o)$ in the timed inertial frame S were adopted as reference characters, with respect to which the characters of the true trip were determined. In other words, the characters of the virtual trip $(B \in S \to O \in S)$ when light is received, are the reference characters. In the scaling transformations of the second type, the characters of the hypothetical trip $(B \in S \to O \in S)$ at the instant of light's emission are adopted as the reference characters.

Formal View

The end points of the hypothetical and the true trips are not the same. In fact the respective end points of these trips are O, o' in S and o, O'in s. Thus, by no means we may talk about absolute length and duration as we had done in the case of the scaling transformations of first type. In spite of this we may understand the relations (13.7) "formally" as defining *new common units of length and time in s and in S* , with

(13.11) $$lsS \equiv lsS(u,\theta) = LS/\Gamma(u,\theta),$$
$$tsS \equiv tsS(u,\theta) = TS/\Gamma(u,\theta),$$

where LS and TS are the geometric units of length and time in the *timed* inertial frame *S*. Setting

(13.12) $$l = l_c.lsS, \qquad t = t_c.tsS\ ;\ L = L_c.LS, \qquad T = T_c.TS,$$

we find using (13.7) and (13.11),

(13.13) $$l_c = L_c = lLen,\ \ t_c = T_c = tTim.$$

Noting the above underlined statements, the last relations mean that *in terms of the common unit lsS (tsS) the reading lLen (tTim) of the length (duration) of the true light trip* $(b\ at\ B \to o \in s)$ *in s and in S, is the same as the reading of the* *geometric length* $\left|\overrightarrow{BO}\right| = R$ $(the\ geometric\ duration\ R/c = T)$ *in the timed frame S using the geometric unit LS (TS).*

Implications of the Euclidean Geometry of the Timed Inertial Frame

We shall abbreviate the "transformation of the second type" by type II. In type II it is possible to view the light trip as occurring conclusively within the timed frame *S*. The pulse emerges from a point $B \in S$ and ends at a moving point $o' \in S$ which was at the instant of light's emission at $O \in S$. But since the geometry of the timed inertial frame S is Euclidean, it should be possible to express the characters of any light trip in *S* in terms of the geometric characters of the hypothetical light trip $(B \to O)$ using the familiar rules of Euclidean geometry. Indeed, writing the transformations (13.8) in the form

$$t = \frac{R}{c}\Gamma(u, \pi - \theta), \qquad r = ct, \tag{13.14}$$

and using the definition of the Euclidean factor given in part IV, we obtain

$$\frac{t}{\sqrt{1-\beta^2}} = \frac{R}{c}G(u,\theta), \qquad r = ct, \tag{13.15}$$

which is the Euclidized form of type II. Thus the period of the true light trip $(b \to o)$ is calculated in two steps

- we employ the rules of Euclidean trigonometry to the triangle BOo' to calculate $r' \equiv ct' \equiv ct/\sqrt{1-\beta^2}$ in terms of the initial radius vector R and the angle $\theta = \angle\left(\vec{R}, \overrightarrow{Oo'}\right)$,
- we multiply $t' = t/\sqrt{1-\beta^2}$ by $\sqrt{1-\beta^2}$ to find the proper period *t*.

In subsequent sections we shall apply type II to explain some well known optical effects. We shall also discuss the possibility of extending the scaling transformations to deal with the case in which only *S* is a timed inertial frame while *s* is uniformly accelerating relative to *S*.